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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematics.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Mathematics and the American Institution of Learning

An examination of the 1937-1938 membership roster* of the American Mathematical Society discloses that of a membership comprising two thousand one hundred thirty-nine (2,139) individuals, one thousand eight hundred thirty-one (1,831) are connected with institutions of learning, that is, with the University, the College, or the High School, while only three hundred eight members are not so connected. Expressed in percentages, 85.6 per cent of the members of the A. M. S. derive their livings from institutions of learning, while only 14.4 per cent do not. Needless to say, nearly all of the 85.6 per cent of membership referred to are connected with the College or the University.

A priori considerations should lead one to expect a check-up of mathematics in England and the nations of Europe to reveal a similar ratio between the number of institutional and the number of non-institutional mathematicians of those countries. But as this note is concerned primarily with the situation in America, such investigation for the present is waived.

In Smith and Ginsberg's *A History of Mathematics in America Before 1900*† is found a fairly complete picture of the humble beginnings of mathematics in the United States. The following quotation partly describes the picture: "As to the eighteenth century—there were no native periodicals devoted to Science, nor were there any learned societies until about the middle of the century. The colleges showed no originality in mathematics,......, and but few individuals produced anything worth mentioning".

In striking contrast with this sterile state of American mathematics throughout the 18th century was the contemporaneous activity of English and European mathematicians—an activity immortally brilliant with such figures as Euler, Lagrange, Laplace, Lambert, Clairaut, Maclaurin, and a host of lesser lights.

*September, 1938 issue of the Bulletin of the American Mathematical Society list of officers and members 1937-1938.

†Number five. The Carus Mathematical Monograh. Published by Mathematical Association of America.

Of course, the cause of this contrast is known to all. Between 1700 and 1800 men and women of the New World were busy building a necessary national civic fabric, and time for mathematics and the sciences was, perforce, small and scattered. But between 1800 and 1900, particularly after 1850, much of the marvelous energy and resourcefulness that had built a great and new republic became released for the establishment and up-building of America's educational fabric. Space is not availble here for recording the evidence that between 1875 and 1900, with a swiftness as amazing as that with which an American republic had been founded, was created by American scholars a body of mathematical literature which shines with a lustre hardly surpassed by that of the mathematics produced in England and Europe during the same period, (see pp. 154-193 of the Smith and Ginsberg Monograph). The fact, that during this time many "sons of wealthy colonists" in America acquired their doctorates in mathematics at European Universities, only emphasizes the quality of an American energy that dared to cross oceans to bring back materials needed in the construction of an American made mathematics.

Just as a period came when men and women in America were too busy building a national civic fabric to give much attention to the erection of colleges; so, later came a time where they were too busy building colleges for the dissemination of the old classic and established cultures, to have much time for the encouragement of mathematical research. But when in 1888 a small group of mathematicians, headed by Thomas Scott Fiske, obeyed an impulse to cooperative research, and organized the first mathematical Society of the United States, it is significant that nearly all of the group held positions on university or college faculties. Among them were Howard VanAmringe, Emory McClintock, Michæl Pupin and Harold Jacoby of Columbia; J. K. Rees of Washington University; W. Woolsey Johnson of U. S. Naval Academy; Simon Newcomb and Thomas Craig of Johns Hopkins; Daniel A. Murray of Cornell; H. B. Fine of Princeton. Thus in a real sense was the *university* the mother of organized mathematical research in America.

The New York Mathematical Society developed into the American Mathematical Society. Of the latter was born later (1916) the Mathematical Association of America. Established and nurtured by these great organizations have come journals serving as carriers and distrib-

utors of mathematical research and exposition. Here again has the American institution of learning borne its generous share of the burden. From the faculty of Drury College came the mathematician who in 1894 founded *The American Mathematical Monthly*, namely Benjamin Finkel. The *American Journal of Mathematics* came into being in 1878, mainly a materialization of the vision of J. J. Sylvester of Johns Hopkins. In 1900 Bryn Mawr, Cornell, Haverford, Princeton, Columbia, Chicago, Yale, Northwestern and Harvard jointly financed the establishment of the *Transactions* of the Society*

Since 1900 have appeared marked evidences that the American institution of learning desires to be more than a mere joint promoter, with the Society, of the interests of American mathematics. It would seem possible that a keener vision has been reached by the University and the College—a vision sensing the essential solidarity of their own institutional interests and the interests of mathematical science. In 1912 Princeton took over the *Annals of Mathematics*, a research periodical. The year 1921 marked the founding of the *Journal of Mathematics and Physics* by Massachusetts Institute of Technology. Yeshiva College of New York City, under the leadership of Jekuthiel Ginsberg, established in 1932 *Scripta Mathematica*, a quarterly journal of mathematics. In the year 1935, Duke University of North Carolina put forth the first issue of a new research quarterly, *Duke Mathematical Journal*. It was in 1935 that Louisiana State University, acting through its president, James Monroe Smith, volunteered a complete financial sponsorship of NATIONAL MATHEMATICS MAGAZINE.

Surely, stronger and stronger must ever become the bond that binds the institution of learning to the Science of Sciences! For, first, more and more do they seem to be led by the same destiny, the destiny of *SERVICE* to Humanity on always widening levels; second, more and more must exact *learning* be interpenetrated with some measure of *mathematics*.

S. T. SANDERS.

*"A History of Mathematics in America Before 1900", by Smith and Ginsberg.

On Cubic Diophantine Equations

By W. V. Parker and A. A. Aucoin
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By use of a multiplicative domain, Carmichael* obtains a solution of the equation

$$(1) \qquad x^3+y^3+z^3-3xyz=u^3+v^3+w^3-3uvw.$$

He says "This solution, so readily obtained, unfortunately lacks generality." By a second method he gets expressions for x, y, u, v in terms of arbitrary parameters a, b, z, w. These are not shown to give the general solution of (1). But he does show† that these expressions give all solutions of $x^3+y^3=u^3+v^3$ by letting $z=w=0$.

In this paper we solve a more general equation which gives the general solution of (1) as a special case. Let $f(x_i)$ denote a homogeneous polynomial of degree three in n unknowns $x_1, x_2, \ldots\ldots\ldots, x_n$ whose coefficients are integers. Suppose that there are n integers $a_1, a_2, \ldots\ldots\ldots, a_n$ not all zero such that

$$\frac{\partial f}{\partial x_j}=0 \text{ for } x_i=a_i \quad (i,j=1,2,\ldots,n).$$

Let $g(y_k)$ be any homogeneous polynomial of degree three in m unknowns $y_1\ y_2, \ldots\ldots\ldots, y_m$ whose coefficients are integers. If in the equation

$$(2) \qquad f(x_i)=g(y_k)$$

we let $x_i=a_is+\alpha_it$, $y_k=\beta_kt$, $(i=1,2,\ldots,n)$, $(k=1,2,\ldots,m)$ we get

$$(3) \quad s^3f(a_i)+s^2t\sum_{j=1}^{n}\alpha_j\frac{\partial f}{\partial a_j}+st^2\sum_{j=1}^{n}a_j\frac{\partial f}{\partial \alpha_j}+t^3f(\alpha_i)=t^3g(\beta_k).$$

Since
$$\frac{\partial f}{\partial a_j}=0 \ (j=1,2,\ldots,n)$$

and by Euler's Theorem

$$3f(a_i)=\sum_{j=1}^{n}a_j\frac{\partial f}{\partial a_j},$$

*Carmichael, *Diophantine Analysis*, (1915), p. 62, formulas (4) and (5).
†*ibid*, page 65, formula (13).

it follows that $f(a_i)=0$, and (3) becomes

$$st^2\sum_{j=1}^{n} a_j\frac{\partial f}{\partial \alpha_j}=t^3[g(\beta_k)-f(\alpha_i)]. \tag{4}$$

If we choose $s=g(\beta_k)-f(\alpha_i)$, $t=\sum_{j=1}^{n} a_j\frac{\partial f}{\partial \alpha_j}$, (4) is satisfied and we get

$$x_i=a_i[g(\beta_k)-f(\alpha_i)]+\alpha_i\sum_{j=1}^{n} a_j\frac{\partial f}{\partial \alpha_j},\quad (i=1,2,\ldots,n) \tag{5}$$

$$y_k=\beta_k\sum_{j=1}^{n} a_j\frac{\partial f}{\partial \alpha_j},\quad (k=1,2,\ldots,m)$$

as a solution of (2) in terms of the $n+m$ arbitrary parameters $\alpha_1\,\alpha_2,\ldots,\alpha_n,\ \beta_1,\beta_2,\ldots,\beta_m$.

By a solution of (2) we shall mean a solution in integers. If $\alpha_1,\alpha_2,\ldots,\alpha_n,\ \beta_1,\beta_2,\ldots,\beta_m$ are integers it is obvious that x_i and y_k as given by (5) are integers. If $x_i=\lambda_i$, $y_k=\mu_k$ is a solution of (2), then $x_i=K\lambda_i$, $y_k=K\mu_k$, where K is an integer different from zero, is also a solution*. We call such solutions *equivalent*. We say that a set of values for x_i, y_k in terms of arbitrary parameters *determines* all solutions of (2) if it is possible to choose the parameters so as to get a solution equivalent to any existing solution

Let us see now if (5) determines all solutions of (2). Suppose $x_i=\lambda_i$, $y_k=\mu_k$ is any solution of (2). In (5) choose $\alpha_i=\lambda_i$, $\beta_k=\mu_k$ and we get $x_i=K\lambda_i$, $y_k=K\mu_k$, where

$$K=\sum_{j=1}^{n} a_j\frac{\partial f}{\partial \lambda_j}\quad ,$$

which is equivalent to the given solution if $K\neq 0$. Therefore (5) determines all solutions of (2) except those which are also solutions of the quadratic equation

$$\sum_{j=1}^{n} a_j\frac{\partial f}{\partial x_j}=0.$$

The condition that (5) determine all non-trivial solutions of (2) depends only on the form of f.

*If $K=0$, the solution is trivial.

We return now to the solution of the special equation (1).

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}=0$$

if $x=y=z=1$, so that (5) becomes

$$
\begin{aligned}
x&=(\lambda^3+\mu^3+\nu^3-3\lambda\mu\nu)+(\alpha-\beta)^3+(\alpha-\gamma)^3\\
y&=(\lambda^3+\mu^3+\nu^3-3\lambda\mu\nu)+(\beta-\alpha)^3+(\beta-\gamma)^3\\
z&=(\lambda^3+\mu^3+\nu^3-3\lambda\mu\nu)+(\gamma-\alpha)^3+(\gamma-\beta)^3\\
u&=3\lambda(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\alpha\gamma-\beta\gamma)\\
v&=3\mu(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\alpha\gamma-\beta\gamma)\\
w&=3\nu(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\alpha\gamma-\beta\gamma).
\end{aligned}
\tag{6}
$$

From the above discussion (6) determines all solutions of (1) except those for which

$$x^2+y^2+z^2-xy-xz-yz=0. \tag{7}$$

We may show that (7) has no solutions other than $x=y=z$. Suppose $x=\alpha$, $y=\beta$, $z=\gamma$ is any integral solution of (7). Then $x=\alpha-1$, $y=\beta-1$, $z=\gamma-1$ is also a solution. There is no loss of generality is supposing that one of the numbers, say α, is positive. By continuing to subtract 1 from the values of x, y, z, we will get a solution in which x is zero and hence y and z must be solution of $y^2-yz+z^2=0$. But this equation has no solutions other than $y=z=0$. So that $y=z=x=\alpha$ is the only solution of (7).

We may say therefore that (6) determines all solutions of (1) except the trivial ones $x=y=z=\alpha$, or $x=\alpha$, $y=\delta$, $z=-(\alpha+\delta)$, $u=v=w=\beta$ or $u=\beta$, $v=\gamma$, $w=-(\beta+\gamma)$. We have shown, incidentally, that all solutions of

$$x^3+y^3+z^3-3xyz=0$$

except the trivial one $x=y=z$, are solutions of $x+y+z=0$. These* may be obtained from (6) by setting $\lambda=\mu=\nu=0$. Or we see by inspection that $x+y+z=0$ is satisfied by $x=\alpha$, $y=\beta$, $z=-(\alpha+\beta)$ where α, β are arbitrary integers.

To get all solutions of (1) for which $w=0$ (or $v=w=0$) it is sufficient to choose $\nu=0$ (or $\mu=\nu=0$) in (6).

*See Dickson, *History of the Theory of Numbers*, Vol. 2, p. 590, ref. 298.

The Solutions of the Quadratic Equation Obtained by the Aid of the Trigonometry

By H. T. R. Aude
Colgate University

Since my school days in Copenhagen, Denmark, I have had, among my books, a copy of Jerome de Lalande's Tables (Holtze, Leipzig). On page 356 there are given certain formulas for solving the quadratic equation $x^2+px=q$ by the aid of trigonometry. Four different cases are considered, according to the signs, plus or minus, of the numbers p and q. And the case, when $p^2<-4q$, is passed by with the mere statement that the roots are imaginary. Recently, looking over this page, the thought occurred that these methods could be combined.

In this paper it is shown that the solutions of the quadratic equation, in all cases, can be obtained by the aid of the trigonometry. The solutions, for the case when the roots are complex, appear at once in an interesting and useful form, the polar form. Furthermore, it seems that if the numerical calculation of the roots is to be done, and if the coefficients of the equation are numbers given to three or more digits, then this method would be more efficacious.

In the general quadratic equation $ax^2+bx+c=0$ we may, without loss of generality, take $a>0$. The cases when b, or c, or both are equal to zero are trivial and will not be considered. The usual form for the solutions of the quadratic is

$$x=(-b\pm\sqrt{b^2-4ac})/2a.$$

This will be rewritten, according as c is positive or negative.

$$x=\sqrt{\frac{c}{a}}\left(-\frac{b}{2\sqrt{ac}}\pm\frac{\sqrt{b^2-4ac}}{2\sqrt{ac}}\right),\quad c>0. \tag{1}$$

$$x=\sqrt{\frac{-c}{a}}\left(\frac{-b}{2\sqrt{-ac}}\pm\frac{\sqrt{b^2-4ac}}{2\sqrt{-ac}}\right),\quad c<0. \tag{2}$$

Consider the last formula first, to dispose of the case when c is negative. Choose an angle α between 0° and 180° such that

$$2\sqrt{-ac}\cot\alpha = -b.$$

Substituting in (2) gives

$$x = \sqrt{\frac{-c}{a}}\,(\cot\alpha \pm \csc\alpha)$$

which written in a form suitable for logarithmic work

$$(2.1) \qquad x_1 = \sqrt{\frac{-c}{a}}\cot\frac{\alpha}{2}, \qquad x_2 = -\sqrt{\frac{-c}{a}}\tan\frac{\alpha}{z},$$

gives the two solutions in the trigonometric form for the case when $c<0$.

Turn to the case where c is positive, see (1), and consider the fractional expression $-b/2\sqrt{ac}$. This fraction may be, numerically, greater than, equal to, or less than 1. If the fraction is improper, or numerically equal to unity, write

$$\csc\alpha = -b/2\sqrt{ac}$$

where α is the principal angle, $-90° \leq \alpha \leq 90°$. The solutions given in (1) become

$$x = \sqrt{\frac{c}{a}}\,(\csc\alpha \pm \cot\alpha)$$

and the solutions in the trigonometric form are

$$(1.1) \qquad x_1 = \sqrt{\frac{c}{a}} \cdot \cot\frac{\alpha}{2}, \qquad x_2 = \sqrt{\frac{c}{a}}\tan\frac{\alpha}{2}.$$

If, however, b is numerically less than $2\sqrt{ac}$, then the discriminant is negative. Change the expression in (1) to a form suitable for complex numbers

$$(3) \qquad x = \sqrt{\frac{c}{a}}\left(-\frac{b}{2\sqrt{ac}} \pm i\,\frac{\sqrt{4ac-b^2}}{2\sqrt{ac}}\right).$$

Then place

$$2\sqrt{ac}\cdot\cos\alpha = -b,$$

where α is the principal angle, $0<\alpha<180°$. The two values of x given in (3) become at once

$$x_1 = \sqrt{\frac{c}{a}}\,(\cos\alpha + i\sin\alpha), \tag{3.1}$$

$$x_2 = \sqrt{\frac{c}{a}}\,[\cos(-\alpha) + i\sin(-\alpha)],$$

and the two complex solutions are in the trigonometric or polar form. All cases have been considered. The solution of the general quadratic equation by the aid of trigonometry has been accomplished.

The preceding will now be applied to a few examples of illustration.

Example 1. Solve the equation $0.184x^2+0.0358x-1.018=0$.

cot α is negative and we have

$$\log|\cot\alpha| = \log|-b/2\sqrt{-ac}| = 8.6166-10.$$

Whence $\alpha = 92°22.1'$.

It follows $\log x_1 = \log\left(\sqrt{\frac{-c}{a}}\cdot\cot\frac{\alpha}{2}\right) = .3533;\ x_1 = 2.256$

$$\log(-x_2) = \log\left(\sqrt{\frac{-c}{a}}\cdot\tan\frac{\alpha}{z}\right) = .3897;\ x_2 = -2.452.$$

Example 2. Solve the equation $0.184x^2-0.2704x-0.3564=0$. Here $\cot\alpha = -b/2\sqrt{ac}$ is positive, α is $62°10'$, and the solutions are $x_1=2.309$, $x_2=-.839$.

Example 3. Solve the equation $384x^2+76.2x+1200=0$. Here $\cos\alpha$ is negative and we have

$$\log|\cos\alpha| = \log|-b/2\sqrt{ac}| = 8.7492-10.$$

Whence $\alpha=93°13'$ and the roots are

$$x_1=1.768(\cos 93°13'+i\sin 93°13')=-0.0992+1.765i$$

$$x_2=1.768\,[\cos(-93°13')+i\sin(-93°13')]=-0.0992-1.765i.$$

Example 4. Solve the equation $384x^2-945x+267.6=0$.

From the relation $-b=2\sqrt{ac}\csc\alpha$ it is seen that α is a positive angle. It turns out that $\log\sin\alpha=9.8315-10$, whence $\alpha=42°43'$. Whence the two roots are $0.8348\cot 21°21.5'$ and $0.8348\tan 21°21.5'$,

or $$x_1=2.461 \quad \text{and} \quad x_2=0.3264.$$

Example 5. Solve the equation $x^2+4x+16=0$.

Here $\cos\alpha=-\frac{1}{2}$, then $\alpha=120°$, and the roots are

$$x_1=4(\cos 120°+i\sin 120°)=-2+2\sqrt{3}\,i$$

$$x_2=4(\cos 240°+i\sin 240°)=-2-2\sqrt{3}\,i.$$

Another form for these two roots is $x_1=4e^{2\pi i/3}$, $x_2=4e^{4\pi i/3}$.

The Mathematics division of the Society for the Promotion of Engineering Education has appointed a committee for the purpose of collecting problems from various scientific and engineering fields. These problems are to be suitable for use in the courses in mathematics taken by freshmen and sophomores in the College of Engineering; and not found in the usual mathematics texts.

Any one having at hand such problems is requested to send them to the chairman, Professor J. W. Cell, Box 5548, College Station, Raleigh, North Carolina.

Notable Points Associated with a Triangle

By J. W. Peters
University of Illinois

1. If the circumcircle of a plane triangle is chosen as the unit circle with center at the origin, the vertices may be designated by the complex numbers t_1, t_2, t_3, where t_i is a complex number of modulus one. The purpose of this note is to point out a seemingly not too well known method for finding in terms of t_1, t_2, t_3, the coordinates of some notable points associated with the triangle. The writer became acquainted with the method through the lectures of Professor Frank Morley. While many of the results obtained by this method have been known for a long time in terms of the pedal triangle, which is similar to the image triangle used here, others have never been published before. Furthermore the method brings to light an underlying unity between the Hessian, Brocard, Beltrami, and other points connected with the triangle.

2. *The image triangle.* The equation of a side of the triangle, that joining t_1 and t_2 for instance, is $x+t_1t_2\bar{x}=t_1+t_2$, where $\bar{x}$ is the conjugate of x. If y is any point in the plane, the image or reflection, x, of the point y in this line is given by $x+t_1t_2\bar{y}=t_1+t_2$. If $s_1=t_1+t_2+t_3$, the images of the point y in the sides of the triangle may be written

$$\begin{aligned} x_1 &= s_1-t_1-t_2t_3\bar{y} \\ x_2 &= s_1-t_2-t_3t_1\bar{y} \qquad (2.1) \\ x_3 &= s_1-t_3-t_1t_2\bar{y}. \end{aligned}$$

The triangle formed by the points x_1, x_2, x_3, is called the image triangle of y with respect to the triangle t_1, t_2, t_3. As various simple restrictions are placed on the image triangle, the point y assumes the positions of notable points associated with the triangle t_1, t_2, t_3.

3. *The image triangle equilateral.* Two triangles, x_1, x_2, x_3, and z_1, z_2, z_3, are directly or positively similar if the determinant $|x_i\,z_i\,1|=0$, $(i=1,2,3)$; and they are inversely or negatively similar if $|x_i\,\bar{z}_i\,1|=0$.*

*Morley and Morley, *Inversive Geometry*, p. 154.

If x_1, x_2, x_3, is equilateral, then either

$$\begin{vmatrix} x_1 & 1 & 1 \\ x_2 & \omega & 1 \\ x_3 & \omega^2 & 1 \end{vmatrix} = 0 \qquad \text{or} \qquad \begin{vmatrix} x_1 & 1 & 1 \\ x_2 & \omega^2 & 1 \\ x_3 & \omega & 1 \end{vmatrix} = 0$$

where ω and ω^2 are the imaginary cube roots of unity. If the values of x_i are substituted in the first of these determinants and the resulting equation is solved for $\bar{y}$, it will be seen that

$$(3.1) \qquad \bar{y} = -(t_1+\omega t_2+\omega^2 t_3)/(t_2t_3+\omega t_1t_3+\omega^2 t_1t_2).$$

We shall designate this point by $\bar{h}_p$. Proceeding in a similar manner with the second determinant, it is seen that

$$(3.2) \qquad \bar{y} = -(t_1+\omega^2 t_2+\omega t_3)/(t_2t_3+\omega^2 t_1t_3+\omega t_1t_2).$$

Call this point $\bar{h}_n$.

The vertices of the triangle, t_1, t_2, t_3, are the roots of the cubic equation $x^3-s_1x^2+s_2x-s_3=0$, where s_1, s_2, s_3 are the usual symmetric functions of the roots. Associated with this equation are the quantities $v_1=t_1+\omega t_2+\omega^2 t_3$ and $v_2=t_1+\omega^2 t_2+\omega t_3$, known as the Lagrange resolvents of the cubic. Their conjugates are $\bar{v}_1=(t_2t_3+\omega^2 t_1t_3+\omega t_1t_2)/s_3$ and $\bar{v}_2=(t_2t_3+\omega t_1t_3+\omega^2 t_1t_2)/s_3$. For future reference we shall set down the following equations connecting the Lagrange resolvents and the symmetric functions.

$$\begin{aligned} v_1v_2 &= s_1^2-3s_2 \\ s_3(v_1\bar{v}_1+v_2\bar{v}_2) &= 9s_3-s_1s_2 \\ (3.3) \qquad s_3^2\bar{v}_1\bar{v}_2 &= s_2^2-3s_1s_3 \\ s_3(v_1\bar{v}_1-\omega v_2\bar{v}_2) &= (\omega-1)(t_1^2t_2+t_2^2t_3+t_3^2t_1-3s_3) \\ s_3(v_1\bar{v}_1-\omega^2 v_2\bar{v}_2) &= (\omega^2-1)(t_1t_2^2+t_2t_3^2+t_3t_1^2-3s_3) \end{aligned}$$

The points h_p and h_n are now given by the equations

$$\bar{h}_p = -v_1/s_3\bar{v}_2 \quad \text{and} \quad \bar{h}_n = -v_2/s_3\bar{v}_1,$$

whence $\qquad h_p = -s_3\bar{v}_1/v_2 \quad \text{and} \quad h_n = -s_3\bar{v}_2/v_1.$

They are known as the Hessian or isodynamic points of the triangle $t_1, t_2\ t_3$. Since $h_p\bar{h}_n=1$, the Hessian points are inverse points with respect to the circumcircle of the triangle t_1, t_2, t_3.

Ordinarily, the Hessian points are defined as the intersections of the Apollonian circles C_1, C_2, C_3, where C_i is a circle passing through t_i

and having the other two vertices t_j and t_k as nverse points. The circles C_i are given by the equations

$$(3.4)\qquad |x-t_2|\cdot|t_3-t_1| = |x-t_3|\cdot|t_1-t_2| = |x-t_1|\cdot|t_2-t_3|,$$

where the vertical bars indicate that the moduli of the quantitie; are to be taken. To verify that h_p satisfies the equation for C_3:

$$|x-t_1|\cdot|t_2-t_3| = |x-t_2|\cdot|t_3-t_1|,$$

substitute $-s_3\bar{v}_1/v_2$ for x. The equation reduces to

$$\frac{|t_1-t_3|\cdot|t_1-t_2|\cdot|t_2-t_3|}{|v_2|}=\frac{|\omega^2|\cdot|t_1-t_3|\cdot|t_1-t_2|\cdot|t_2-t_3|}{|v_2|}$$

Since $|\omega^2|=1$, h_p lies on the circle. Similarly it may be verified that h_p lies on C_1 and C_2 and also that h_n lies on the same three circles.

An algebraist would probably define the Hessian points of the triangle as the zeros of the Hessian of the cubic $x^3-s_1x^2+s_2x-s_3=0$. The Hessian of this cubic is

$$(3s_2-s_1^2)x^2+(s_1s_2-9s_3)x-(s_2^2-3s_1s_3)$$

which, by means of the equations (3.3), may be written

$$v_1v_2x^2+s_3(v_1\bar{v}_1+v_2\bar{v}_2)x+s_3^2\bar{v}_1\bar{v}_2.$$

The zeros of this function are the quantities designated by h_p and h_n.

4. *The image triangle similar to the original triangle.* If the image triangle x_1, x_2, x_3, is directly similar to the triangle t_1, t_2, t_3, where the points are chosen in those respective orders, the determinant

$$|x_i \quad t_i \quad 1| = 0 \qquad (i=1,2,3)$$

reduces to

$$s_3\bar{y}\cdot|t_i \quad 1/t_i \quad 1| = 0.$$

The determinant $|t_i\ 1/t_i\ 1|$ is proportional to the area of the triangle t_1, t_2, t_3, and is consequently not zero unless two of the points coincide, which case we shall exclude. Thus $y=0$, the circumcenter. The image triangle of the circumcenter then is directly similar to the original triangle.

If the triangle x_1, x_2, x_3, is directly similar to the triangle t_2, t_3, t_1, we have

$$\begin{vmatrix} s_1-t_1-t_2t_3\bar{y} & t_2 & 1 \\ s_1-t_2-t_3t_1\bar{y} & t_3 & 1 \\ s_1-t_3-t_1t_2\bar{y} & t_1 & 1 \end{vmatrix} = 0$$

*Peters, *American Journal of Mathematics*, Vol. LI, p. 599.

and $$\bar{y}=v_1v_2/(t_1t_2^2+t_2t_3^2+t_3t_1^2-3s_3).$$

By use of (3.3), we have

$$\bar{y}=(\omega^2-1)v_1v_2/s_3(v_1\bar{v}_1-\omega^2v_2\bar{v}_2) \tag{4.1}$$

and $$y=(\omega-1)s_3\bar{v}_1\bar{v}_2/(v_1\bar{v}_1-\omega v_2\bar{v}_2).$$

Call this point B_1.

If the triangle x_1, x_2, x_3 is directly similar to the triangle t_3, t_1, t_2, we find that

$$y=(\omega^2-1)s_3\bar{v}_1\bar{v}_2/(v_1\bar{v}_1-\omega^2v_2\bar{v}_2). \tag{4.2}$$

Call this point B_2.

The Brocard points of a triangle are the points x and z defined as follows:

$$\angle xt_1t_2 = < xt_2t_3 = \angle xt_3t_1 \tag{4.3}$$

$$\angle zt_1t_3 = < zt_3t_2 = \angle zt_2t_1.$$

In the first case the equality may be written*

$$\frac{a-b}{\bar{a}-\bar{b}} \Big/ \frac{c-b}{\bar{c}-\bar{b}}$$

$$\frac{x-t_1}{t_1t_2(\bar{x}-1/t_1)}=\frac{x-t_2}{t_2t_3(\bar{x}-1/t_2)}=\frac{x-t_3}{t_3t_1(\bar{x}-1/t_3)}$$

or $$\frac{t_3x-t_1t_3}{s_3\bar{x}-t_2t_3}=\frac{t_1x-t_1t_2}{s_3\bar{x}-t_1t_3}=\frac{t_2x-t_2t_3}{s_3\bar{x}-t_1t_2}$$

By the theory of composition and division for proportions, we have

$$\frac{(t_3+\omega t_1+\omega^2t_2)x-(t_1t_3+\omega t_1t_2+\omega^2t_2t_3)}{(t_2t_3+\omega t_1t_3+\omega^2t_1t_2)}$$

$$=\frac{(t_3+\omega^2t_1+\omega t_2)x-(t_1t_3+\omega^2t_1t_2+\omega t_2t_3)}{(t_2t_3+\omega^2t_1t_3+\omega t_1t_2)}.$$

If this equation is expressed in terms of the Lagrange resolvents and solved for x, it is found that $x=B_1$. In a similar fashion it may be shown that $z=B_2$. Hence the points B_1 and B_2 are the Brocard points of the triangle t_1, t_2, t_3.

*The angle abc is given by the amplitude of the quotient

5. *The image triangle inversely similar to the original triangle.* For triangle x_1, x_2, x_3 inversely similar to t_1, t_2, t_3

$$|x_i \quad 1/t_i \quad 1| = 0. \qquad (i=1,2,3)$$

This condition reduces to

$$s_3\bar{y}\cdot|1/t_i \quad 1/t_i \quad 1| + |t_i \quad 1/t_i \quad 1| = 0.$$

Since the area of the original triangle is not zero, we conclude that y is the point at infinity.

If the triangle x_1, x_2, x_3 is inversely similar to t_2, t_3, t_1, we find that

$$y = s_3(v_1\bar{v}_1 - \omega^2 v_2\bar{v}_2)/(\omega^2-1)v_1v_2. \tag{5.1}$$

Call this point g_1.

If the triangle x_1, x_2, x_3 is inversely similar to t_3, t_1, t_2, we find that

$$y = s_3(v_1\bar{v}_1 - \omega v_2\bar{v}_2)/(\omega-1)v_1v_2. \tag{5.2}$$

Call this point g_2. The points g_1 and g_2 are the inverses of the Brocard points B_1 and B_2 respectively in the circumcircle. They are called the Beltrami points of the triangle.*

6. By placing other conditions on the image triangle other notable points may be obtained. For instance, if the centroid of the image triangle is y, then y is the symmedian point of the original triangle. If the vertices of the image triangle are on the circumcircle of the original triangle, y is the orthocenter of t_1, t_2, t_3. If the image points lie on a straight line, that line is on the orthocenter of t_1, t_2, t_3, and y lies on the circumcircle of that triangle. It is left to the reader to discover in similar fashion other points associated with the triangle t_1, t_2, t_3.

*Morley and Morley, *Inversive Geometry*, p. 78.

Humanism and History of Mathematics

Edited by
G. Waldo Dunnington

Note on Roman Numerals

By Allen A. Shaw
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The Roman numerals and symbols are well-known to the Europeans and Americans. Still there are a good many *variant forms* of numbers written in the Roman system which are not generally known as well as certain *rules* of formation of numbers by this system. The present writer is in possession of photographs of an excellent chapter on Roman numerals published in *The Tutor's Assistant—A Compendium of Arithmetic and a Complete Question Book* by Francis Walkingame, London, 1845, and obtained through the courtesy of Dr. J. W. Clarson, Dean of the College of Education, University of Arizona.

The Roman numerals are as follows: I, V, X, L, C, D, M. The rules by which the Romans expressed all (small) numbers by these symbols are as follows:

(*a*) When a symbol is repeated the number is repeated also. Thus III means 1 plus 1 plus 1, *not* 111. So XXX is 10 plus 10 plus 10, or 30; CCC is 300.

There is no need for VV (10), LL (100), DD (1000). They are never used as there are separate symbols for these.

A symbol is not usually repeated more than three times. IIII is permissible for 4 but is unusual. Such forms are better written by (*c*) below.

(*b*) When a symbol is placed *to the right* of a greater symbol, the values of the two are *added*.

Thus in VII the two I's to the right of V are added to V so that VII means 7. So XIII is 13, XXII is 22, CCXI is 211, MDCCCXXIII is 1823.

(*c*) When a *single* symbol (I, X, C), is placed *to the left* of a greater symbol, its value is to be *subtracted* from that of the greater.

(For rare cases see photographs at the end of the article when more than one symbol is subtracted at a time).

Thus in IV the I is to the left of V and is to be subtracted from it. So that IV means 5 minus 1, *i. e.*, 4 XC means 100 minus 10, *i. e.*, 90. XCIX means 99.

Not more than one symbol is ever to be subtracted from a greater symbol, but see *rare* forms in the photographs at the end of this paper. Thus IIV is *not* a correct form for 3.

(*d*) A horizontal line over a symbol multiplies its value a thousand times.

Thus $\overline{\text{V}}$ means 5000; $\overline{\text{XX}}$ means 20,000; $\overline{\text{L}}$, 50,000; $\overline{\text{C}}$, 100,000; $\overline{\text{D}}$, 500,000; etc.

For more details the reader is recommended to read *History of Mathematics*, Vol. 2, by D. E. Smith, pages 54 to 63.

The Romans wrote the first 10 numbers thus: I, II, III, IV (IIII), V, VI, VII, VIII, IX (VIIII), X.*

After these ten numbers the Romans arranged others as far as possible on the same model: XI, XII, XIII, XIV, XV.

Thus he writes XIX *in preference* to IXX for 19.

Thus the *tens* are X, XX, XXX, XL (or XXXX), L, LX, LXX, LXXX, XC, C, etc.

For reasons just named he writes XCIX for 99, rather than IC. Following the analogy of XIX, the correct sign for 1900 is MCM rather than MDCCCC. The latter form, however, is the one employed in inscriptions.

CDXXXIV is 400 (CD), 30 (XXX), and 4 (IV), *i. e.*, 434. CMXLIX is 949; MDCCCXCIX is 1899; MCMXXXVIII is 1938.

*The reader will notice here traces of the use of a *quinary scale* of notation. Note also that while the Roman system of numeration is a decimal system, that of the notation is not.

A SYNOPSIS OF THE ROMAN NOTATION.

In compiling the following Roman characters, the editor has been chiefly indebted to the following works: — *Peter Bungus*, Bergomatis Numerorum, &c., second edition, 4to., Bergomi ∞ IↃXCI.; *M. I. Tritheme*, Polygraphie, 4to. Paris. 1561; and *I. Gerrard*, Siglarium Romanum, 4to. London, MDCCXCII.

1 = I.	8 = VIII, or IIX.
2 = II.	9 = IX, or VIIII.
3 = III.	10 = X.
4 = IV, or IIII, or Iv.	11 = XI.
5 = V, or Λ.	12 = XII.
6 = VI.	13 = XIII.
7 = VII.	14 = XIV, or XIIII.

From "The Tutor's Assistant"

15 = XV, or $\overset{v}{X}$.
16 = XVI.
17 = XVII.
18 = XVIII, or IIXX, or XIIX.
19 = XIX, or XVIIII.
20 = XX, or ⨯⨯ or ⨯⨯.
30 = XXX, or $\overset{x}{III}$, or ⨯⨯⨯.
40 = XL.
45 = VL, or XLV.
50 = L, or Γ, or λ
60 = LX, or $\overset{xx}{III}$.
70 = LXX.
80 = LXXX, or XXC, or $\overset{xx}{IIII}$.
81 = ⨯⨯ CI.
90 = XC, or LXL.
99 = LXXXXIX, or XCIX, or IC.
100 = C, or [, or $\overset{c}{I}$.
200 = CC, or ƆƆ, or $\overset{c}{II}$.
300 = CCC, or $\overset{c}{III}$.
400 = CCCC, or CD, or $\overset{c}{IV}$.
500 = D, or IƆ, or $\overset{c}{V}$, or Ð.
600 = DC, or IƆC, or $\overset{c}{VI}$.
700 = DCC, or IƆCC, or $\overset{c}{VII}$.
800 = DCCC, or CCM, or IƆCCC, or $\overset{c}{VIII}$, or CC∞.
900 = DCCCC, or CM, or IƆCCCC, or C∞, or $\overset{c}{IX}$.
1,000 = M, or $\overset{o}{M}$, or ∞, or Φ, or Dɑ, or CIƆ, or $\overline{I}$, or 8, or ✕, or M, or ⋔, or ✕.
1,100 = MC, or CIƆC, or ∞C.
2,000 = MM, or IIM, or $\overline{II}$, or II∞, or ∞∞, or CIƆCIƆ, or ✕✕.
3,000 = MMM, or IIIM, or III∞, or $\overline{III}$, or CIƆCIƆCIƆ, or ∞∞∞, or ✕✕✕.
4,000 = MMMM, or IVM, or IV∞, or $\overline{IV}$, or CIƆIƆƆ, or ∞IƆƆ.
5,000 = $\overline{V}$, or V∞, or VM, or IƆƆ, or ⊾, or $\overset{M}{V}$, or I + ƆƆ, or VCIƆ, or ⊍, or Ɒ.
6,000 = $\overline{VI}$, or VI∞, or VIM, or IƆCM, or IƆƆCIƆ, or IƆƆ∞.
7,000 = $\overline{VII}$, or VIIM, or IƆƆMM, or VII∞, or VIICIƆ, or IƆƆCIƆCIƆ, or IƆƆ∞∞.
8,000 = $\overline{VIII}$, or VIIIM or IƆƆMMM, or VIII∞, or CIƆCIƆCCIƆƆ, or ∞∞CCIƆƆ.
9,000 = $\overline{IX}$, or IXM, or IX∞, IƆƆMMMM, or CIƆCCIƆƆ, or ∞CCIƆƆ
10,000 = $\overline{X}$, or XM, or CCIƆƆ, or X∞, or CMƆ, or ⋔, or Ψ, or CC·I·ƆƆ, or C·C·I·Ɔ·Ɔ, or $\overline{X}$ or ⋔, or ⊍, or Φ.
20,000 = $\overline{XX}$, or XX∞, or XXM, or CC$\overline{I}$ƆƆCCIƆƆ.
50,000 = $\overline{L}$, or L∞, or IƆƆƆ, or LM, or ⊾.
100,000 = $\overline{C}$, or CM, or C∞, or CCCIƆƆƆ, or ⋔, or ⋔, or ⋔, or ⋔, or ⋔, or ⋔.

101,000 = $\overline{\text{CI}}$, or CCCIↃↃↃM.
200,000 = $\overline{\text{CC}}$, or CCM, or IICCCIↃↃↃ, or $\overline{\text{II}}$.
500,000 = $\overline{\text{D}}$, or DM, or D∞, or IↃↃↃↃ.
900,000 = $\overline{\text{DCCCC}}$, or $\overline{\text{CM}}$.

1,000,000 = $\overset{\bar{c}}{\text{X}}$, or $\overline{\text{M}}$, or XCM, or CCCCIↃↃↃↃ, or

2,000,000 = $\overset{\bar{c}}{\text{XX}}$, or $\overline{\text{MM}}$, or XXCM.

5,000,000 = $\overset{\bar{c}}{\text{L}}$, or LCM.

9,000,000 = $\overset{\bar{c}}{\text{XC}}$, or LXXXXCM.

10,000,000 = $\overset{\bar{c}}{\text{C}}$.

90,000,000 = $\overset{\bar{c}}{\text{DCCCC}}$.

100,000,000 = $\overset{\bar{c}}{\overset{c}{\text{X}}}$, or XCCm.

156,000,000 = $\overline{\text{CLVIM}}$.

800,000,000 = $\overset{\bar{c}}{\overset{c}{\text{LXXX}}}$.

1,000,000,000 = $\overset{\bar{c}}{\overset{c}{\text{C}}}$.

10,000,000,000 = $\overset{\bar{c}}{\overline{\text{C}}}$, or XCCCm.

Note 1. It sometimes happens, that when a number is expressed by more than two characters, it is distinguished from the character on the left hand by a point: thus, 140 = C.XL; 499 = CD.XC.IX; 1500 = CIↃ.IↃ, or ∞.D; 1,582 = CIↃ.IↃ.XXCII; 1,699 = CIↃ.DC.IC; 1,718 = M.IↃCC.XIIX; 1,750 = CIↃ.IↃCC.L; 1,494 = M.$\overset{o}{\text{C}}$CCC.LXL.IIII; 1,697 = M.IↃC.XC.VII; 2,000 = CIↃ.CIↃ; 3,000 = CIↃ.CIↃ.CIↃ; 7,000 = IↃↃ.∞.∞; 8,833 = IↃↃMMM.IↃCCC.XXX.III; 73,000 = IↃↃↃ.CCIↃↃ.CCIↃↃ.CIↃ.CIↃ.CIↃ; 67,489 = $\overline{\text{LXVII}}$.CCCC.LXXXIX; 1,620,829 = $\dot{\text{X}}$VI.XX.DCCC.XXIX.

Note 2. Some authors denote ten by a small line drawn across another, *thus* †, and five by a similar small line not drawn across but only touching the other, *thus* ├; in this manner they have ‡ for 15. A small line placed after the character (I) to the right denotes that unity is added; *thus*, ├| signifies five and one, or 6; †| denotes 11; ‡| 16; ‡|| 17; and so on. Again, by a cypher, or 0, standing apart denotes a hundred, a 0 placed after a number, at the top, denotes that the number is to be taken 100 times; *thus*, 0, or I° denotes one hundred; 00, or II°, two hundred; 000, or III°, three hundred; ├° five hundred; †° ten hundred or one thousand. These marks are frequently seen added to magical characters. (*See* H. C. Agrippa, *De Occulta Philosophia*. Folio, 1533.)

The following Roman characters, found in ancient Latin manuscripts, are seemingly arbitrary: 1 = V; 6 = Q; 10 = D; 11 = O, or XIM;

30=~~666~~, or ~~ZZZ~~; 40=F, or P, or Q; 50=K; 51=k; 70=S; 80=R; 90=N, or NA; 120=IXX; 150=Y; 160=T; 200=$\overline{\text{C}}$C, or H; 250=E; 300=B; 400=G, or P; 500=A, or Q; 900=NA; 1000=CXↃ, or IXI, or CLↃ, or MS, or MIL; 1500=M+L+, or ML; 2000=Z; 2200=IICC; 5000=ICC, or QↃ; 10,000=CC·I·CC, or ↃMC, or ↃMↃ, or IↃↃ, or IMI; 11,000=$\overline{\text{O}}$; 40,000=$\overline{\text{P}}$; 50,000=DMIↃↃↃ, or ML, or D+, or ↃMC; 70,000=$\overline{\text{S}}$; 80,000=$\overline{\text{R}}$; 90,000=$\overline{\text{N}}$; 100,000=C+ML+, or CI, or $\overline{\text{CM}}$, or LCM, or C, or C+; 150,000=$\overline{\text{Y}}$; 160,000=$\overline{\text{T}}$; 200,000=CC+MM+; 400,000=$\overline{\text{G}}$; 500,000=D+, or $\overline{\text{Q}}$, or qMD, or qↃ, or DMIↃↃↃ, or CQↃ; 1,000,000=CMↃ, or CqↃ, or CqↃ, or $\overline{\text{MS}}$, or $\overline{\text{MM}}$; 2,000,000=$\overline{\text{Z}}$.

TABLES OF ENGLISH COIN, WEIGHTS, MEASURES, &c.

An Abstract of the late Act for ascertaining and establishing Uniformity of Weights and Measures, which passed in June 1824, *and commenced operation on January* 1. 1826.

1. The *Standard Yard* is declared to be the distance between the centres of the two points on the gold studs in the straight-brass rod, now in the custody of the clerk of the House of Commons, executed by Mr. Bird, a celebrated optician, whereon is engraved "*Standard Yard*, 1760," the brass being at the temperature of 62° by Fahrenheit's thermometer. It is to be called the "*Imperial Standard Yard*," from which all other measures of extension, whether lineal, superficial, or solid, are to be divided as formerly. (For its divisions and multiples see Tables VII., IX., X., and XI., pages 33 to 36.)

2. The *Imperial Standard Yard*, if lost, defaced, or otherwise injured, may be restored by comparing it with the pendulum * vibrating seconds of mean time, in the latitude of London, in a vacuum on the

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* As a *pendulum* (or a small leaden ball suspended by a *very fine* thread), when made to vibrate in *very small* arcs, affords a ready means of measuring small portions of time; and as a pendulum vibrating seconds, in addition to its other important uses, forms the most unexceptionable *universal standard of measures*, the following particulars respecting that instrument are subjoined, for the inspection of the inquisitive student.

The times of vibration of pendulums of different lengths are as the square roots of their lengths; and the number of vibrations made in a given time is in the inverse ratio of the square roots of their lengths: *consequently* the lengths of pendulums are as the squares of the times of a vibration; or in the inverse ratio of the squares of the number of vibrations made in a given time.

Examples.

Ex. 1. If the length of a pendulum vibrating seconds, or 60 times in a minute, is 39·1393 inches; in what time will a pendulum 12 inches long vibrate, and how many vibrations will it make in a minute?

As $\sqrt{39{\cdot}1393}$ *in.* : $\sqrt{12}$ *in.* :: 1 *sec.* : ·5537 second. Also, 12 *in.* : 39·1393 *in.* :: 60² *times* : 108·356² *times*; or, $\sqrt{12}$ *in.* : $\sqrt{39{\cdot}1393}$ *in.* :: 60 *vib.* : 108·356 *vibrations.*

Ex. 2. If the length of a pendulum vibrating seconds is 39·1393 inches, what is the length of a pendulum vibrating half seconds, or 120 times a minute?

As 1² *vib.* : ($\frac{1}{2}$)² or 120² *vib.* : 60² *vib.* :: 39·1393 *in.* : 9·7848 inches.

Note. The lengths of *pendulums vibrating seconds* vary, with the force of gravity, in different latitudes, and also in the same latitude at different elevations above the level of the sea.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES McGIFFERT

The Subject or the Student

By GLENN JAMES
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When a teacher comes before a class of young people who are ambitious to advance, but as yet are not exactly sure what advancement means, he faces a most sacred responsibility. If he has not come with forethought, he will have forced upon him suddenly the necessity of deciding whether he will teach the subject, as such, or teach the students how to get it for themselves, whether he will act as a stamp collector or a dynamo builder, whether the student exists for the subject or the subject for the student, whether the system makes the man or the man makes the system. Practice based on the former philosophy we choose to call quantitative teaching, that based on the latter qualitative. The extremes of these two viewpoints are opposed in a sense similar to that which differentiates between Fascism and Democracy. The incentives which lead in either direction are much the same. On the one hand we have efficiency in the accomplishment of an immediate end; on the other a slower, more laborious progress toward a less obvious but much more desirable goal.

One can cover much more ground by just telling his class all about it. It is temptingly easy and simple to hand them the ready made facts directly, especially in mathematics where subject matter is pretty clearly defined. All one needs to do is to get up a set of lectures or choose a textbook that contains the best proofs and discussions, whatever best may mean, present them to class after class and require the classes to copy them in the former case and reproduce them in either, after, of course, he has explained away their difficulties, then at the end of the semester find out how full of what, they have become. This entire process is indeed so simple and so efficient that enough ground can be covered to make both class and teacher feel that the class has made great progress, and overlook the fact that it has been in the direction of uniformity and mediocrity.

On the other hand, if one does qualitative teaching and treats his students as subjects to be taught to learn, his problem becomes very much more complex. He has as many subjects as students, each with an approach to his problems that is attuned to his individualistic style of thinking, which itself is a function of his disposition and his previous individualistic experiences. Standardized procedure is no longer a first objective. Modified proofs and discussions, better or worse, but produced by the student in his own, perhaps eccentric, way are at a premium. The teacher becomes fused with the growing personalities and maturing minds of the class. Sample demonstrations are largely his part of the subject matter work. Suggestions, Socratic questioning, and the inspiring of enthusiam are his heavy but happy duties. The question of how much the student can learn is replaced by how much he can develop, not as a conformer to set patterns, but as a designer of new ones.

The classic objection to this procedure is that there are certain fundamental facts that the student must have as tools before he can go ahead with subsequent work. This objection seems to miss the point. The question at issue is whether the student shall acquire these tools himself, with as little guidance as possible, or have them pressed upon him.

Such qualitative teaching as the above assumes that the abilities to learn and to discover can be developed, and makes their development the central objective of all class methods; while the other sort of teaching either assumes that these abilities are fixed quantities or that development will come as a by-product of the presentation of the subject matter. The latter is, of course, true to a limited extent, determined mostly by the amount of home work they do.

I shall not try to define "abilities to learn and discover." Whether they mean some inherited capacity of this kind or that would not alter the fact that in college the good students are those who have some time or other been encouraged to develop themselves and the poor ones are those who have been discouraged or at least not encouraged, and that mere cramming with subject matter will not make a good student out of a "flunker", while improvement of his methods of study and development of his ability to learn will do so. Twenty-five years ago, nearly everyone thought mathematicians were "born" and could not be made. Now we know they can and are being made. This places a heavier responsibility upon teachers, but makes our work more sacred and more thrilling.

Most teachers have the qualitative philosophy, but usually practice the quantitative method. This is partly due to the criticisms

they meet when off the trodden path, and partly to the fact that the quantitative system is self-perpetuating. Generally, a student must know such a mass of facts in order to carry his next courses that one is almost forced into cramming them into him in the quickest way. This situation arises from the fact that courses are laid out mostly on the basis of their factual content, which is true from the first grade well into graduate work, less marked, however, in the lower grades and in the last two years of graduate work. The "No Man's Land" in this matter is the high school and undergraduate college courses. By the time students get to college they have been so trained that most of them support the teacher who is a veritable Santa Claus, who "gets his subject across" so that they can pass his final examinations. The fact that they are pursuing a mirage does not occur to them unless and until they get into some kind of life-work other than repeating these facts to the next generation in this endless rote chain, erroneously called education.

The great need for more qualitative teaching is generally recognized, but progress in that direction is not going to be made soon by reducing the subject content of courses. The tendency now is decidedly in the opposite direction. However, there is an alternative, on the whole more desirable. That is to reduce severely the size of classes. The obstacle in the way of this procedure is limited finances. If the public, especially parents, who ultimately control the purse strings of our schools, understood what the teachers could do with classes of not more than six or five students, they would see to it that money for financing their schools on that basis was put first on the public "must" list and that the teachers then taught their pupils instead of the subject. But the people are not going to finance education adequately for a long, long time. They are going to keep on wasting many times the money this would require on wars and other futile causes, probably because they are more ostentatious and immediate than education. Meanwhile we teachers must struggle on with our oversized, content-crowded courses, never surrendering our ideals, no matter how difficult they are to attain.

An expedient that I have found very helpful under existing conditions is to go very slowly as to subject matter during the first weeks of the semester, making this a sort of fruition period in which subject matter is used merely as an instrument with which to develop self-confidence and method of attack. By the end of three or four weeks a class has usually gotten in "shape", a state quite obvious in itself and always indicated by a spontaneous pick-up in the rate at which subject matter is being covered. It is then not difficult, with the

additional power that has been gained, to catch up with the classes which distribute their subject matter uniformly throughout the semester. One is always handicapped somewhat in carrying out this plan by quantitative teachers in other courses putting on pressure by means of excessive assignments. But this can be offset to a considerable extent by stirring up enthusiasm and interest in the recitations and putting more and more responsibility on the students for what work is assigned.

Some time can be saved by telling the class at the beginning all about the plan and showing them how they can speed it up by volitional effort. Indeed, without some explanation, progress will be obstructed by those who, not understanding your objectives, think you are wasting time. On the other hand, one's explanation usually "takes" on one or two students who in turn become a great help in developing the others. They act as a sort of leaven in the group.

At the end of the fruition period one by no means throws over all qualitative work and turns to "cramming". Most of the students of the class should by then be in shape to pursue their work largely on their own initiative. Those who have not responded can be given special attention during recitations, or met individually outside to the extent of the instructor's time.

I have found these non-responders to consist, in the main, of two types: those who are very set in their habits of imitating, committing theorems, processes, etc., and in general following very strictly the trodden paths; and those who are extremely underconfident. The best wholesale treatment for the former that I have found is individual assignments of problems that they cannot solve, integration problems that result in elliptic integrals, for instance. The underconfident are the worst taught group in college. I have dealt with them heretofore in this magazine so will only suggest here that most of this class will improve when sent to the blackboard day after day with problems, very easy ones at first and harder ones later, but never hard enough to stall them until they have gained a lot of self-confidence.

(*The Cause and Cure of Delinquency in College Mathematics*, Vol. XI, No. 6, March, 1937, NATIONAL MATHEMATICS MAGAZINE.)

Another Start

By Brother Bernard Alfred, F. S. C.
Manhattan College, New York City

As a teacher of mathematics I read with interest Mr. Edwin Olds' article, *A Fresh Start,* in the March issue of the National Mathematics Magazine. If the lesson the author describes were the first one given to his new class, and it seems to me it was, he covered too much matter; he left too many terms undefined; he took too much for granted. I have presumed to offer to the readers of the National Mathematics Magazine the technique I would employ in teaching Mr. Olds' class and the method I use in teaching a similar topic to my students.

I do not teach repeaters separately but have them begin anew with students who never had the subject before. Hence, my lessons, as described below, may be a little too elementary for a class of repeaters. However, I find it a good practice to assume that they know nothing about the subject and act accordingly. Such students are generally difficult to teach for many reasons. The principal ones are, their natural slowness in learning and their mental attitude, "This is a tough subject; I cannot get it." Consequently, it seems to me, if I had Mr. Olds' class I would first remark that I will go slowly and explain things carefully so that every one will be able to understand them. I would encourage the class to ask questions if an explanation is not clear, or to request that the explanation be repeated. Having put them at ease by such friendly and encouraging words I would then talk about Calculus; telling them of its applications, mentioning a few in detail; acquainting them with some of its history and letting them know how it has aided civilization, both materially and intellectually. Briefly, I would strive to arouse in them a desire to know something about so marvelous a branch of human knowledge, a tactic Mr. Olds neglected to use. Thus would the first hour end.

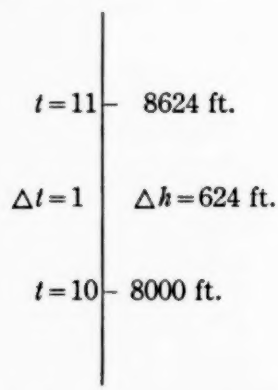
My second lesson would recall to the class that Calculus is something they really want to know. But before actually beginning any work in it we shall behave like the husband who puts coal in his cellar in the fall to use when winter comes, by studying a few preliminary notions which will be necessary for later use. Hence, I would review and even explain constants, arbitrary and absolute; variables, dependent and independent; functions like $y=mx+b$, $y=ax^2+bx+c$, and

$y=f(x)$; pausing, as occasion may suggest, to ask and answer questions. The second lesson would end having explained $y=f(2)$ and $y=f(a)$, and having assigned some written work.

The third lesson would begin with a review of the preceding one, recalling especially the meaning of $y=f(x)$ and $y=f(a)$. This leads easily and naturally to the symbols Δx and Δy, the meanings of which must be absolutely clear to every student, since they are of paramount importance in arriving at the definition of derivative and in deriving the formulas of the differential Calculus.

I have had more success in approaching the definition of derivative by considering a problem on a projectile shot vertically upward than by any other method. It is more interesting than the geometrical approach and appears to the dull or repeating pupils to be more natural. Consequently, after having taught the meaning of Δt and Δh, I illustrate their use in a problem similar to the following: A bullet is fired straight upward. Its height h ft. after t sec. is $h=960t-16t^2$. How far did it travel from the 10th to the 11th second?

The students have no trouble finding the height at the 10th and the 11th seconds to be 8000 ft. and 8624 ft., respectively. I use a diagram similar to the accompanying figure to illustrate the meaning of the problem and our results. They have no difficulty understanding that $\Delta t=1$ sec. and $\Delta h=624$ ft.



I would now ask "What is the average speed of the bullet from the 10th to the 11th sec.?"

A large number promptly answer 624 ft./sec.

"Is that the speed at the 10th sec.?" "At the 11th sec.?"

Many answer No correctly. For those who do not seem to understand that No is correct I enlighten them by using the familiar example: If a car travels 100 miles in four hours, how fast does it travel?

It is obvious to the slowest student that 25 mi./hr. is only average rate and does not tell him the rate at the second or third hours, etc. It is now a simple matter to show that $\Delta h/\Delta t$ represents the average speed over a certain distance for a certain interval of time.

The stage is set to solve the first problem in Calculus. I tell my class that the next time we meet we shall answer the question, "What is the exact speed at the 10th sec.?" The lesson and discussion ends by assigning for exercise the above problem requiring the pupils to find the average speed of the bullet from the 10th to the 12th sec., and from the 49th to the 50th sec. They leave the class looking forward to the next meeting. Several of the interested students, not always the best, come to the desk and want to know the exact speed at the 10th sec. I tell them that that is a secret to be revealed later. They are disappointed but smile pleasantly. I can never suppress a little chuckle on an occasion like this.

The fourth lesson is a crucial one. I had promised the class that we would determine the exact speed at the 10th sec. but that can be found only by considering the limit of $\Delta h/\Delta t$ as Δt approaches zero and I have not as yet even mentioned the word limit. Besides, the notion of limit is as important in Calculus as it is difficult for the repeating student to grasp. Hence, this lesson must be exceptionally well planned and I hope will be exceptionally well taught.

At the beginning of the lesson I would inform the class that I can find the average speed from $t=10$ to $t=12$, by writing $t=12$ as $10+\Delta t$, and at the end let $\Delta t=2$. Many do not see why I should use such a complicated method when the one we had in the last session was so much easier to use and to understand. I explain carefully, with the aid of a diagram, that by using Δt I can find the average speed of the bullet not only from $t=10$ to $t=12$ by letting Δt equal 2, but the average speed for any other interval of time by letting Δt equal the proper value. This generality, I explain, will be a real aid to us in finding the exact speed at the 10th sec. We proceed to solve the problem as follows: (details are omitted).

1. When $t=10$ the height is 8000 ft.

2. When $t=10+\Delta t$, the height is $8000+640\Delta t-16\overline{\Delta t^2}$.

3. Therefore, $\Delta h=640\Delta t-16\overline{\Delta t^2}$.

4. The average speed is $\dfrac{\Delta h}{\Delta t}=\dfrac{640\Delta t-16\overline{\Delta t^2}}{\Delta t}$

   or $\Delta h/\Delta t=640-16\Delta t$

5. Letting $\Delta t=2$, we get $\Delta h/\Delta t=608$ ft./sec., the average speed from $t=10$ to $t=12$.

I let $\Delta t=1$, .5, .1, .01, .001, and solve for corresponding values of $\Delta h/\Delta t$.

Now the real teaching begins! I would ask "Which is the best approximation to the speed of the bullet at the 10th sec.?" All agree that the value obtained when $\Delta t=.001$ is the best. I would ask "Why?" The answer is obvious to most of the class. "What have I been doing to Δt?" Obviously making it smaller. It is also apparent that the smaller Δt becomes the better will be the approximation to the exact speed at the 10th sec. I would now ask the obvious question "What is the exact speed at the 10th sec.?" Practically every one immediately recognizes that the answer must be 640 ft./sec. I have kept my promise and feel the students understand very well what I taught them. The lesson is now almost over. At this time some pleasant chap invariably raises his hand and asks "What is the exact speed at the 11th sec.?" I answer "That is your home work for tonight." I quickly review the lesson, at the same time telling the students how to do their exercise, as follows:

1. Find the height when $t=11$ sec.
2. Find the height when $t=11+\Delta t$ sec.
3. Find the change in height from $t=11$ to $t=11+\Delta t$.
4. Divide by Δt to get average speed.
5. Find the limit as Δt approaches 0. That is the exact speed at the 11th sec.

The fifth lesson would see $t=11$ become $t=t$ and $t=11+\Delta t$ replaced by $t=t+\Delta t$ and the derivative defined.

The method described above takes very little for granted and it arouses the interest of the students for Calculus. A little blackboard work will remove many of the difficulties that remain in the back of the slower student's mind. Thus, would I launch Mr. Olds' repeaters again into the mysteries of the Calculus.

Mathematical World News

Edited by
L. J. Adams

Frederick Wood, head of the department of mathematics at the University of Nevada, has been made Dean of the College of Arts and Science at the same institution.

Mr. Everett Harris has been appointed instructor in mathematics at the University of Nevada.

Miss Margaret Jensen has been appointed assistant in mathematics at the University of Nevada.

The annual meetings of the Mathematical Association of America, American Mathematical Society, American Association for the Advancement of Science and the National Council of Teachers of Mathematics were scheduled to be held at Richmond and Williamsburg, Virginia on December 27-31, 1938.

We would like to call the attention of our readers to the international periodical *Compositio Mathematica* published by P. Noordhoff in Groningen, Netherlands. It is edited by L. E. J. Brouwer, Th. De Donder, H. Hopf, G. Julia, J. M. Whittaker and others. The complete editorial board includes forty-seven members. It is designed to promote the development of mathematics through the publication of research papers, and at the same time to further international co-operation. Authors of accepted manuscripts receive one hundred reprints free of charge.

At the recent meeting of the Academy of Sciences in Moscow, several mathematical papers were presented, including:

1. *Some new estimations of the analytical theory of numbers.* I. M. Vinogradow.
2. *A theorem of O. Schmidt.* D. Koliankowsky.
3. *Some theorems on singular integrals.* I. Natanson.
4. *Mean values of measurable functions.* B. Lewitan.

Professor Tibor Rado's article on *Subharmonic Functions,* published in *Ergebnisse der Mathematik und ihrer Grenzgebiete* (No. 1, 1937)

and written in English, should prove of interest to American research mathematicians who have failed to see it thus far.

The Royal Academy of Brussels met on August 1, 1938 and among the research papers presented were:

1. *Researches on the cyclic involutions belonging to an algebraic surface.* L. Godeaux.
2. *Subharmonic functions.* M. Brelot.
3. *Multiple Abelian curves.* P. Defrise.

The three hundred fifty-fifth meeting of the American Mathematical Society was held at the University of California at Los Angeles, November 26, 1938. There were eleven research papers presented and seven papers presented by name alone.

The London Branch of the Mathematical Association of England will hold its annual business meeting on Saturday, January 28, 1938 at Bedford College. Professor G. B. Jeffery, president of the branch, will preside. At that time there will be a general discussion of the Geometry Report. The February meeting will be devoted to a consideration of the project method of teaching mathematics.

The British Film Institute publishes a monthly bulletin on films. Among the films recently reviewed in this bulletin was one on the differential equation

$$\frac{d^2x}{dt^2}+x=A\sin Nt.$$

This attempt to describe a mathematical problem by cine-diagrams should prove of interest to American mathematicians. It is the belief of the editor of this department that very little has been done (in America) with this method of teaching mathematics.

The Victoria Branch of the Mathematical Association of England held meetings on March 23, April 26, June 7, July 27, August 31 and September 20, in 1938. Some of the interesting features of these meetings were: Mr. W. S. Dickson's talk on *Complex Angles,* Dr. Mildred Barnard's paper on *Statistics in Experimental Research,* Professor H. R. Hamley's address on *The Teaching of Post-primary Mathematics* and Professor Cherry's lecture on *Newton's Principia—1687 and 1937.*

Mathematicians interested in the mechanical integration of differential equations will want to examine the article on this subject

by D. R. Hartree in the October, 1938 issue of *The Mathematical Gazette* (London, England).

At the joint meeting of the Mathematical Association of America and the National Council of Teachers of Mathematics to be held in Williamsburg, Virginia on December 30, 1938 the addresses will be:

1. *A College Mathematics Teacher Views Teacher Training.* A. A. Bennet.
2. *The Professional Preparation of Mathematics Teachers.* F. L. Wren.
3. *Mathematics in the Training of Arithmetic Teachers.* R. L. Morton.

Bibliography and Reviews

Edited by
P. K. SMITH and H. A. SIMMONS

Advanced Mathematics for Engineers. By H. W. Reddick and F. H. Miller. John Wiley and Sons, New York, 1938. x+473 pages. $4.00.

A quotation from the preface to *Advanced Mathematics for Engineers* is the surest means of indicating the authors' purpose: "The present book has been evolved from courses given by the authors to juniors and seniors in civil, electrical, mechanical, and chemical engineering at the Cooper Union Institute of Technology. These courses were designed to show some of the various roles played by advanced mathematics in engineering technology."

The mathematical content of the text is much the same as that of the classic texts in advanced calculus. The distinguishing feature of this book is the emphasis on physical applications and the ability to use the mathematics as well as to understand the fundamental mathematical principles. How well this is done may be illustrated by reference to the applications of differential equations given in the first chapter. Careful discussions are there made of the parabolic reflector, problems in dynamics, chemical solutions, heat flow, electrical circuits, deflection of beams, harmonic motion, damped and forced vibrations. With each application is found a statement of the engineering principles required, the derivation of the associated differential equation, and finally the solution of the engineering problem.

This method of treatment is adhered to throughout the text. Proofs are given where the proof is not too difficult and will aid in the application of the theory. If the proof is too difficult or if its inclusion would serve no further useful purpose, the student is referred to some source where the proof may be found.

This text contains more than enough material for a year's advanced course. The teacher could choose not only from the mathematics material but also from those applications which are most likely to be of interest and use to his immediate class.

The stimulation of such a book as this is apparent. Certainly the authors have fulfilled their objective of teaching the student not only some advanced mathematics but also where and how to use it. This is consistent with the trend in teaching engineering mathematics

courses—the trend toward teaching mathematics and at the same time its use.

About seventy-five percent of the engineers get enough mathematics with the completion of a year's course in calculus. The other twenty-five percent need more mathematics and physics courses. Some will need graduate work in these fields, but all need such a course as this book contains. Moreover, engineering students will develop much more rapidly if their study of mathematics (as well as physics) is presented in an engineering framework. Aware of this need, and convinced of the merits of this book, this reviewer would sincerely recommend its use for such students.

North Carolina State College. J. W. CELL.

Isaac Newton, 1642-1727. By J. W. N. Sullivan. Macmillan, New York, 1938. xx+275 pages. $2.50.

For his Newton biography, Sullivan has leaned heavily on previous biographers such as Brewster, Brodetsky and Dean L. T. More. He has not adduced any hitherto unknown information about Newton. The author used freely Newton's correspondence and records of his acquaintances. This biography has the virtue of being extremely readable and, though brief, gives a clear, well-rounded portrait and interpretation of Newton the man, even though the reader may not always agree with the author's conclusions. His mathematical and physical explanations are not too technical for the general reader.

This biographer felt that one of the most remarkable things about Newton's career was the very sudden flowering of his scientific genius. Sullivan's book is built around the central theme that Newton was extraordinarily indifferent towards his own genius, and there seems to be much valid evidence to back up this theory. An important element of the book is the discussion of the influence of the Newtonian thought-world on subsequent science. This is asserted to be due in part to the great scientist's possession of "physical insight" of the first order.

It seems to this reviewer that Sullivan overstressed Newton's alleged view "that science was relatively unimportant" (as contrasted with theology and alchemy) in explaining the fact that he was comparatively unproductive during the last forty years of his life. This view was not different from that of many other educated men of Newton's time and cannot be called a paradox. It is too severe to state that he was a mere figure-head during the second half of his life.

Sullivan rightly brings out, however, that Newton was "human" in his personal and official behavior. He gives on the whole a concise, well-balanced account of Newton's calculus controversy with Leibniz.

Attention is properly directed to Newton's characteristic trait of keeping his work to himself rather than publish it immediately and have his quiet disturbed by controversy. This attitude has been present in many other great scientists. "His life was one long meditation, but his interest in the subject of his meditations was exhauted in the act of understanding it." Sullivan felt that Newton was one of the most fascinating characters of whom we have any record, and this biography exemplifies such a point of view.

The greatest question in connection with Newton's career will always be: Why did he leave Cambridge where he had made his name immortal, desert science, and go to the Mint where he made a fortune and enjoyed the social pleasures of London and the royal court? Was it because he felt that his genius had run its course or was it because his stipend at Cambridge was too small? Sullivan seems to explain this by means of the very human trait quoted above. Perhaps he is correct. Perhaps Newton's intense meditations had exhausted him physically and he merely desired a change.

It is a singular fact that there is no definitive edition of Newton's collected works, correspondence, and works. A new biography of Newton ought to stir up interest in such a project. The biography under review unfortunately has no index, no illustrations, and no chapter titles. It is reasonably free of misprints; the name of Lagrange is misspelled on page 258. Sullivan died (August 11, 1937, aged 51) a few days after it was completed and his friend, Charles Singer, who saw it though the press and made minor verbal corrections, has appended a twenty page biographical memoir of the author. He states that the book was more than ten years in preparation and owes much to the inspiration and encouragement of Eddington.

State Teachers College, G. WALDO DUNNINGTON.
La Crosse, Wisconsin.

The Collected Works of George Abram Miller. Volume II. The University of Illinois Press, Urbana, 1938. xi+557 pages.

The University of Illinois continues its great service to the mathematical world in honoring both itself and one of its foremost scholars by the publication of the second volume of Professor G. A. Miller's collected works. This volume is uniform in appearance with the first

which was published in 1935 and retains the same high quality of typography and editorial workmanship. Opposite the title page is a picture of Professor Miller in 1910, a date slightly outside the period covered by the present volume but sufficiently close to portray the author as he appeared at that time.

"This volume contains Professor Miller's contributions to the theory of finite groups during the years 1900 through 1907." It contains 109 memoirs—the preface by the Editorial Committee erroneously states 108. The first and last of these were written by Professor Miller especially for the present volume. The remaining 107 appeared in 18 American and foreign periodicals, a number which in itself testifies to the ubiquitous interest in Professor Miller's work.

During the period covered by this volume there was increased emphasis on abstract groups and this altered point of view is reflected herein. A large proportion of the monographs in Volume I were concerned with the determination of all substitution groups of a given degree. This problem and the companion one of determining all abstract groups of a given order were attacked by Professor Miller with consummate skill in his earlier papers. In the volume under review there are still papers dealing with these topics, for example, (88) which was written in collaboration with G. H. Ling, wherein the intransitive substitution groups of degree eleven are listed, and (99) wherein all groups of order 168 are determined. But the main emphasis is on the properties of the group of isomorphisms, on the properties of groups which satisfy certain conditions, and on the problem of groups generated by operators satisfying certain conditions.

Memoir (93) is concerned with "Groups defined by the orders of two generators and the order of their product." Groups which are generated by two operators had been studied by Hamilton in 1856, and in 1882 in the *Mathematische Annalen* Dyck considered the additional conditions that must be satisfied by the generators of a group when the orders of these generators and the order of their continued product are given and the group is to be of finite order. Professor Miller shows in this memoir that if l,m,n are the orders of three substitutions L, M, N, such that one of these substitutions is the product of the other two, then: *If two of the three numbers l, m, n are equal to 2, or if $l=2$, $m=3$, and $n=3$, 4, or 5, the group generated by L and M is completely defined by the values l, m, n. For all other sets of values of l, m, n any two of the substitutions L, M, N can be so constructed as to generate any one of some infinite system of groups of finite order.* In this article Professor Miller set the stage for many important investigations by himself and others in a field which is still productive of significant results.

In addition to the memoirs of a pure'y technical character which will appeal to the specialist in group theory there are some fifteen or more of an expository or historical nature which the general reader will find worth while. These include, among others, two book reviews, the second and third reports on recent progress in the theory of groups of finite order, the report on groups of infinite order, several articles which deal with the use of the concept of group in elementary mathematics, and the two memoirs which were written especially for the volume under review.

The first of these is entitled "Note on the history of group theory during the period covered by this volume" and "aims to present those developments which have close contact with the writer's own work without exhibiting some equally important results which are less closely connected therewith." To the specialist in group theory it will give an appropriate setting for the technical papers of the volume as Professor Miller looks back on them over a period of thirty years; to the general reader it will give an interesting story of the development in the first part of the present century of an important field of mathematics.

The second is entitled "Primary facts in the history of mathematics" and is the concluding article of the volume. "During these years Professor Miller began to write on subjects in the general history of mathematics, and his interest in general mathematical history has continued," says the preface by the Committee, and this last article contains a statement of the philosophy underlying his writings along historical lines and of his judgment regarding important elements in the development of mathematical history. During forty years Professor Miller has been untiring in his efforts to emphasize the importance of accuracy in historical writing. "Statements that can be neither proved nor disproved are not historical," "Statements should be tested by their implications," "In the history of mathematics errors are an irritant but truth is a stimulant" are quotations which illustrate the spirit of his historical writings. The general reader will find this memoir most interesting.

There is an appendix in which are listed forty papers published between 1900 and 1907 and not included in this volume. "Some of these were omitted because they are covered by papers which are included; the rest are concerned with history or are elementary illustrations, applications, or expositions." Doubtless the reasons for omission are good, nevertheless it does seem unfortunate that all of them could not have been included in full in their appropriate places. It is always possible to read a mathematician's publications in their

original setting where they are a part of general mathematical literature and must be viewed in that light. But in collected works, each in its appropriate place, they throw light not only upon the development of the subject of which they treat, but also upon the development of the man himself, upon the development of a personality, and upon the maturing of an intellect. This is perhaps one of the greatest functions of collected works, and every memoir, however unimportant, plays a part therein. However, the University of Illinois has been too generous in what it has done for one to be critical that it did not do more. One can only close by saying that the book is a very worthy tribute to a mathematician who has greatly enriched an important field of mathematics.

University of Illinois. F. E. JOHNSTON.

Problem Department

Edited by
Robert C. Yates and Emory P. Starke

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to Robert C. Yates, College Park, Md.

SOLUTIONS

No. 15. Proposed by *W. E. Byrne*, Virginia Military Institute.

From Granville-Smith-Longley's *Calculus*, p. 113:

"What is the minimum value of

$$y = ae^{kx} + be^{-kx} \ ? \qquad \text{Ans. } 2\sqrt{ab}."$$

It happens that the answer is correct only under certain assumptions. What are these assumptions and what occurs in the other cases where $ab \neq 0$?

Solution by *W. Irwin Thompson*, Los Angeles City College and *Albert B. Farnell*, Louisiana State University.

We have: $y' = k(ae^{kx} - be^{-kx})$; $y'' = k^2(ae^{kx} + be^{-kx})$, with critical values given by

$$ae^{kx} - be^{-kx} = 0 \quad \text{or} \quad e^{kx} = \sqrt{b/a}.$$

1. If $a > 0$, $b > 0$, $k \neq 0$, then $e^{kx} = \sqrt{b/a}$ yields $y = 2\sqrt{ab}$ with y'' positive. Thus a minimum.
2. If $a < 0$, $b < 0$, $k \neq 0$, then $y = 2\sqrt{ab}$ with y'' negative at $e^{kx} = \sqrt{b/a}$. Thus a maximum.
3. If $ab \leq 0$, $k \neq 0$, the function is monotonic and no critical points exist.
4. If $k = 0$, the function reduces to $y = a + b$ which is trivial.

Accordingly, the conditions under (1) must be fulfilled to produce a minimum.

Also solved by the *Proposer*.

No. 59. Proposed by *A. W. Randall*, Prairie View, Texas.

Find the volume of a 2-inch square hole cut through a sphere 12 inches in diameter, the axis of the hole being the diameter of the sphere.

Solution by *C. A. Balof*, Lincoln College, Illinois.

Let V be the required volume.

$$V=8\int_0^1\int_0^1\int_0^{(36-x^2-y^2)^{1/2}} dz\ \ dy\ \ dx$$

$$V=4\int_0^1 (35-x^2)^{\frac{1}{2}}dx+4\int_0^1 (36-x^2)\text{ arc sin }(36-x^2)^{-\frac{1}{2}}\,dx.$$

The second integration is performed by integrating by parts. Then

$$V=2(34)^{\frac{1}{2}}(638/3)\text{arc sin } 35^{-\frac{1}{2}}-4\int_0^1 (36-\tfrac{1}{3}x^2)x^2dx/(36-x^2)(35-x^2)^{\frac{1}{2}}$$

$$V=2(34)^{\frac{1}{2}}(638/3)\text{arc sin } 35^{-\frac{1}{2}}-4\int_0^1 x^2dx/(35-x^2)^{\frac{1}{2}}$$

$$-(8/3)\int_0^1 x^4dx/(36-x^2)(35-x^2)^{\frac{1}{2}}.$$

The first integral offers no difficulty. In the second integral let $z^2=35-x^2$. Then

$$V=4(34)^{\frac{1}{2}}+(428/3)\text{arc sin } 35^{-\frac{1}{2}}+(8/3)\int_{\sqrt{35}}^{\sqrt{34}} (35-z^2)^{3/2}dz/(1+z^2)$$

$$V=4(34)^{\frac{1}{2}}+(428/3)\text{arc sin } 35^{-\frac{1}{2}}-(8/3)\int_{\sqrt{35}}^{\sqrt{34}} (35-z^2)^{\frac{1}{2}}dz$$

$$+96\int_{\sqrt{35}}^{\sqrt{34}} (35-z^2)^{\frac{1}{2}}dz/(1+z^2)$$

In the last integral let $z=35^{1/2}\sin\Theta$. Then

$$V=8(34)^{1/2}/3+(428/3)\text{arc sin } 35^{-1/2}-(428/3)\text{arc sin }(34/35)^{1/2}$$
$$+214\pi/3+3456\int_{\pi/2}^{\text{arc sin}\sqrt{34/35}} d\Theta/(1+35\sin^2\Theta).$$

In the last integral let $\tan\Theta/2=x$. Then

$$V=8(34)^{1/2}/3+(428/3)\text{arc sin } 35^{-1/2}-(428/3)\text{arc sin }(34/35)^{1/2}$$
$$+214\pi/3+6912\int_1^{(\sqrt{35}-1)/\sqrt{34}}(1+x^2)dx/(1+142x^2+x^4)$$

$$V=8(34)^{1/2}/3+(428/3)\text{arc sin } 35^{-1/2}-(428/3)\text{arc sin}(34/35)^{1/2}$$
$$+214\pi/3+[6912(6+35^{1/2})/12]\int_1^{(\sqrt{35}-1)/\sqrt{34}} dx/[x^2+71+12(35)^{1/2}]$$
$$+[6912(6-35^{1/2})/12]\int_1^{(\sqrt{35}-1)/\sqrt{34}} dx/[x^2+71-12(35)^{1/2}].$$

Upon integrating and simplifying the result, we have

$$V=8(34)^{1/2}/3-(428/3)\text{arc sin } 33/35-650\pi/3+576\text{ arc tan } 6(34)^{1/2}$$

$V=47.56$ in.^{3} approximately.

No. 232. Proposed by *V. Thébault*, Le Mans, France.

In the system of numeration with base 11, form a perfect square of six digits having each of the forms *abcabc* and *aabbcc*.

Solution by *Fred Marer*, Los Angeles, California.

All numbers will be written in the system with base 11; the digit *ten* will be represented by X.

(A) $abcabc=abc\cdot 1001=abc\cdot 4\cdot 9\cdot 34$ implies that abc is the product of 34 by a one- or two-digit square. The only squares sufficiently small are 04, 09, 15, 23. Hence $abc=125$, 283, 499, or 771 and the required squares are 125125, 283283, 499499, and 771771.

(B) We note two cases: (1) $c=0$, and (2) $c\neq 0$.

(1) $aabb00=a0b\cdot 1100=a0b\cdot 4\cdot 3\cdot 100$ implies that $a0b$ is three times a square. The squares to consider are 45, 59, 74,..., 334. If

these be multiplied by 3; 201, 300 and 804 are the only resulting products having the necessary form. Thus $aabbcc=221100$, 300000, or 884400.

(2) Suppose $(rs1)^2=aabbcc$. The units digit of the square of $rs1$ is 1, preceded by $2s$ or $(2s-10)$. Since this digit is also 1, we have $s=6$. Since $aabbcc=a0b0c\cdot 11=a0b0c\cdot 3\cdot 4$, 6 must be a divisor of $rs1$; but $10\equiv -1 \pmod 6$, hence $rs1\equiv r-s+1\equiv r-6+1\equiv 0 \pmod 6$ and thus $r=5$. However $(561)^2=289311$ is not of the required form.

Similarly we test $rs2$, $rs3$, etc. There appear in all twelve possibilities for the square root $(c\neq 0)$, viz. 561, 512, 473, $X73$, 864, $5X5$, 606, 347, 947, 738, 699, $64X$. These give the only six-digit squares, divisible by 6 and with last two digits duplicates. But only $606^2=336633$ has its remaining digits of the required form.

If distinct digits are required for a, b, c, then the solutions are: (A)125125, 283283; (B) 221100, 884400.

Also solved by *C. W. Trigg* and the *Proposer*.

No. 233. Proposed by *E. P. Starke*, Rutgers University.

Find all sets of four distinct integers, having no common divisor greater than 1 and such that each is a divisor of the sum of the other three.

Solution by the *Proposer*.

Let x, y, z, w be the required integers. We may put

$$x+y+z=(d-1)w \qquad x+z+w=(b-1)y \tag{1}$$

$$x+y+w=(c-1)z \qquad y+z+w=(a-1)x,$$

where a, b, c, d are integers. From (1) we obtain immediately

$$x+y+z+w=ax=by=cz=dw, \tag{2}$$

and then, if $ax\neq 0$, $y=ax/b$, $z=ax/c$, $w=ax/d$. Thus we have $x+ax/b+ax/c+ax/d=ax$ or

$$1/a+1/b+1/c+1/d=1. \tag{3}$$

We may also put (3) in the alternate form

$$[(ab-a-b)c-ab]\,[(ab-a-b)d-ab]=a^2b^2. \tag{4}$$

Before proceeding we note the following trivial cases: (*i*) If (x,y,z,w) is a solution, so is $(-x,-y,-z,-w)$ a solution. (*ii*) If $ax=0$, whence $x+y+z+w=0$, (x,y,z,w) is an obvious solution. (*iii*) If

$x+y+z=0$ and w is any common multiple of x, y, z; then (x,y,z,w) is a solution provided we accept the statement, "0 is divisible by w." (This corresponds to $d=1$.)

Henceforth we consider only cases not included under (*ii*) and (*iii*).* We next establish the result: *Of the letters in* (3), *three, at least, are positive; and the smallest positive letter equals* 2. This is easily seen from the following facts: If $c<0$ and $d<0$, from (3) we have $1/a+1/b=1/(-c)+1/(-d)>1$, a result not possible unless $a=1$ or $b=1$ as in (*iii*). If $d<0$ and $2<a<b<c$, we have $a\geqq 3$, $b\geqq 4$, $c\geqq 5$, and hence $1=1/a+1/b+1/c+1/d\leqq 1/3+1/4+1/5+0=47/60<1$. If $2<a<b<c<d$, we have similarly $1=1/a+1/b+1/c+1/d\leqq 1/3+1/4+1/5+1/6=57/60<1$.

With $a=2$, *the next smallest positive letter, say* b, *must be* 3 *or* 4. The proof of this is also implied in (3); supposing $b\geqq 5$, $c\geqq 6$ and either $d<0$ or $d\geqq 7$, we would have $1\leqq 1/2+1/5+1/6+1/d$ which is impossible for the stated values of d.

Finally we put $a=2$, $b=3$ into (4) to obtain $(c-6)(d-6)=36$. Taking $c<d$, we must have $c-6=1, 2, 3, 4, -9, -12, -18$, or -36, from which (c,d) are found to be (7,42), (8,24), (9,18), (10,15), $(-3,2)$, $(-6,3)$, $(-12,4)$, $(-30,5)$. The requirement that $a\neq d$, etc., eliminates the fifth and sixth of these pairs. Again, if $a=2$, $b=4$, (4) becomes $(c-4)(d-4)=16$, so that (c,d) are given by (5,20), (6,12), $(-4,2)$, $(-12,3)$. Here, the third makes $a=d$, while the fourth is essentially a duplicate of one of the former set.

Our final values for the integers without common divisor, (x,y,z,w) are then (1,6,14,21), (1,3,8,12), (1,2,6,9), (2,3,10,15), $(-1,3,4,6)$, $(-1,6,10,15)$, (1,4,5,10) and (1,2,3,6) in addition to results implied in (*i*), (*ii*), and (*iii*).

Included in the above argument is the solution of the following problem: *find four integers, the sum of whose reciprocals is unity.*

No. 234. Proposed by *D. C. Duncan*, Compton Junior College, California.

(1) If the base and side of an isosceles triangle are relatively prime integers, prove that the internal bisectors of the base angles cannot be integers.

(2) Find the smallest integral isosceles triangle whose three internal bisectors are all integers.

(3) Obtain all triangles whose three sides and three internal bisectors are simultaneously rational.

*i. e. $a,b,c,d\neq 0,1$.

Solution by *E. P. Starke*, Rutgers University.

In the triangle ABC, let m be the length of the internal bisector of angle C intercepted within the triangle. It is not difficult to show that*

$$m=(2ab\cos C/2)/(a+b), \tag{1}$$

with analogous formulas for the other bisectors.

To show result (1), let $c=b$ and a be relatively prime integers. Then, upon squaring (1) and replacing $\cos^2 C/2$ by $(a+2b)/4b$, we have $m^2=a^2b(a+2b)/(a+b)^2$. Let p be any prime divisor of $(a+b)$. Since a and b are relatively prime, p is not a divisor of a, nor of b, nor of $(a+2b)$. Hence m^2 is not an integer, and, *a fortiori*, m is not.

If all sides and bisectors are to be rational, we see from (1) that we must have rational values for $\cos A/2$, $\cos B/2$, $\cos C/2$. Hence also it is easy to show that† $\sin A/2$, $\sin B/2$, $\sin C/2$ are rational. Conversely, if $\sin A/2$, $\cos A/2$, $\sin B/2$, $\cos B/2$ are rational and one side is rational, all necessary conditions for rational sides and bisectors‡ are fulfilled. Now if $\sin A/2$ and $\cos A/2$ are rational, the familiar solution for $u^2+v^2=1$, viz. $u=2rs/(r^2+s^2)$, $v=|r^2-s^2|/(r^2+s^2)$, holds, and we must have $\cos A/2=2rs/(r^2+s^2)$, $\sin A/2=|r^2-s^2|/(r^2+s^2)$, or with these values reversed.

To solve (2), take $B=C$, $\sin B/2=3/5$, $\cos B/2=4/5$, (i. e., $r=2$, $s=1$); then $a=14b/25$ and the bisectors of angles A and B have the lengths $24b/25$ and $112b/195$, respectively. These will be integral if we take $b=c=975$; then $a=546$ and the bisectors are 936 and 560, respectively.§ This is easily seen to be the smallest such triangle on account of the conditions on $\sin B/2$ and $\cos B/2$.

If all possible choices for $A/2$ and $B/2$ and, say, side c are made as above, we shall have all triangles whose sides and internal bisectors are simultaneously rational. For example, if we take $B/2$ as given above for (2), and $\sin C/2=5/13$, $\cos C/2=12/13$, we have a scalene triangle meeting the requirements of (3). If in addition we make $b=169$ the triangle has integral sides ($a=154$, $c=125$) and rational internal bisectors (2600/21, 30800/279, 48048/323). Any triangle

*Let the bisector of angle C meet AB in F. Then $\overline{AF}^2=m^2+b^2-2bm\cos C/2$ and $\overline{BF}^2=m^2+a^2-2am\cos C/2$ and $AF/BF=b/a$. Elimination of AF and BF gives (I).

†e. g. $\sin A/2=\{(s-b)(s-c)/bc\}^{\frac{1}{2}}=(a/s)\{s(s-b)/ac\}^{\frac{1}{2}}\cdot\{s(s-c)/ab\}^{\frac{1}{2}}=(a/s)\cos B/2\cos C/2$.

‡$\sin C/2=[\cos(A/2+B/2)]$ and $\cos C/2$ are rational; thus $\sin C=[2\sin C/2\cos C/2]$, $\sin A$ and $\sin B$ are rational. Thus the other sides are rational and by (I) the bisectors are rational.

§Noted by Fuss, 1813. See L. E. Dickson's *History of the Theory of Numbers* Vol. 2, p. 210, where virtually the above method is suggested—but Cunliffe's example, there cited is incorrect.

which meets the demands of (3) will have also rational external angle bisectors, rational area, rational altitudes, and five rational radii.*

No. 235. Proposed by *C. E. Springer*, University of Oklahoma.

The direction of a gun on a horizontal plane may lie anywhere within a cone of semi-vertical angle Θ, the axis of the cone being fixed at an angle α with the horizontal. If the initial speed of the projectile, v_0, is constant, find the equation of the curve which bounds the field of fire on the horizontal plane. Show that according as

$$\Theta \gtreqless \left|\frac{\pi}{2} - 2\alpha\right|,$$

there is a bi-tangent to the curve perpendicular to the vertical plane of fire, or a tangent to the curve at a point in the vertical plane of fire, or no tangent to the curve perpendicular to the plane of fire.

Note. The case in which Θ is so small that its powers higher than the first can be neglected, is given by MacMillan, *Theoretical Mechanics*, (1927), Exercise 16, page 263.

Solution by *E. B. Wedel*, Holdenville (Oklahoma) High School.

With the origin of coordinates at the gun, let OA, the axis of the cone, lie in the xz plane at an angle α with the x-axis. Let the direction cosines of OA be λ_1, O, ν_1, and those of the generating line OP, which makes an angle Θ with OA, be λ, μ, ν.

The equations of motion of the projectile are $mx''=my''=0$, $mz''=-mg$ with the initial conditions: at $t=0$, $x=y=z=0$. Hence we have the parametric equations of the surface within which all projectiles will move:

$$x=v_0\lambda t, \quad y=v_0\mu t, \quad z=-gt^2/2+v_0\nu t.$$

Since we are interested in the curve within which the projectiles all strike the ground, we let $z=0$ and have the time of flight: $t=0$ and $2v_0\nu/g$.

From the given conditions we have:

$$\lambda^2+\mu^2+\nu^2=1 \text{ and } \cos\Theta=\lambda\cos\alpha+\nu\sin\alpha \text{ or } \lambda=\sec\alpha(\cos\Theta-\nu\sin\alpha).$$

Thus, since $v_0^2=2gh$, we find:

$$(1) \qquad \begin{cases} x=4h\nu\sec\alpha(\cos\Theta-\nu\sin\alpha) \\ y=\pm 4h\nu\sec\alpha\sqrt{\cos^2\alpha-\cos^2\Theta+2\nu\sin\alpha\cos\Theta-\nu^2}, \end{cases}$$

*See solution of Problem E331 soon to appear in the *American Mathematical Monthly*.

where the parameter is ν. The elimination of this parameter gives rise to an equation of the fourth degree. The sum of the squares yields

$$x^2+y^2=16h^2\nu^2(1-\nu^2) \tag{2}$$

which indicates that the curve must lie within a circle whose radius is $4h\nu(1-\nu^2)^{\frac{1}{2}}$.

From (1) we have: $dx/d\nu=4h(\sec\alpha\cos\Theta-2\nu\tan\alpha)$ which has the root $\nu=\frac{1}{2}\csc\alpha\cos\Theta$, the condition necessary for a tangent parallel to the $y-$axis. The corresponding values of x and y are:

$$\begin{cases} x=h\cos^2\Theta/\sin\alpha\cos\alpha, \\ y=\pm(h\cos\Theta/\cos\alpha\sin^2\alpha)\sqrt{(\sin 2\alpha+\cos\Theta)(\sin 2\alpha-\cos\Theta)}. \end{cases}$$

Thus there is a bi-tangent to the curve perpendicular to the vertical plane of fire when $\sin 2\alpha>\cos\Theta$ (i. e., when $\Theta>|\pi/2-2\alpha|$) since y has two real values for every value of x. When $\Theta=|\pi/2-2\alpha|$, then $\sin 2\alpha=\cos\Theta$ and $y=0$. In this case there is a tangent to the curve. When $\Theta<|\pi/2-2\alpha|$, then, for every value of x, y becomes imaginary and there exists no tangent to the curve perpendicular to the plane of fire.

Also solved by the *Proposer* who notes that the right member of equation (2) is a maximum for $\nu=1/\sqrt{2}$ and, therefore, for every choice of α and Θ, the projectile will not fall outside the circle $x^2+y^2=4h^2$. The rectangular equation of the curve is

$$(x^2+y^2\sin^2\alpha+2hx\sin 2\alpha)^2\sec^2\Theta+16h^2(x^2+y^2)(\cos^2\Theta-\sin^2\alpha)$$
$$-8h\sin 2\alpha\cdot x(x^2+y^2+hx\cot\alpha)=0,$$

with $x-$intercepts $2h\sin 2(\alpha-\Theta)$ and $2h\sin 2(\alpha+\Theta)$.

No. 236. Proposed by *V. Thébault*, Le Mans, France.

Form a perfect square of nine digits for each of the forms

*aabbccdd*5 and 5*qqrrsstt*.

Solution by *C. W. Trigg*, Los Angeles City College.

I. Put $N^2=5qqrrsstt$. Then

$$22360<N<24500. \tag{1}$$

Since every square may be put in the form $(50k\pm a)^2$, $a=1,2,\ldots,25$, whose last two digits are the same as those of a^2, inspection of the squares

from 1^2 to 25^2 shows that the only possible values of t are 0 and 4. Also the last non-zero digit of N^2 is 4. Thus

(2) for $t=4$, $N=50k\pm12$;

for $t=0$ and $s=4$, $N=500k\pm120$;

for $t=s=0$ and $r=4$, $N=5000k\pm1200$.

Since $10\equiv-1 \bmod 11$, we must have $N^2\equiv5 \bmod 11$, and hence

(3) $$N\equiv\pm4 \bmod 11.$$

Of the eighteen numbers satisfying (1), (2), and (3), the following are found to have squares of the required form: $(23338)^2=544662244$, $(22620)^2=511664400$, and $(23800)^2=566440000$.

II. Put $M^2=aabbccdd5$. The same argument as in I, (1) and (3) leads to the results

(4) $$d=2,\ M=50k\pm15 \quad \text{and}$$

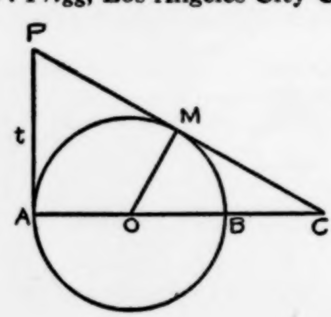
(5) $$M\equiv\pm4 \bmod 11.$$

If now $a=1$, $11\cdot10^7<M^2<12\cdot10^7$ and thus $10488<M<10955$. The numbers between these limits which satisfy (4) and (5) are 10685 and 10765. Analogously for $a=2,3,\ldots,9$, we find thirteen more possibilities. Four of these have squares of the required form, viz. $(14835)^2 = 220077225$, $(15165)^2 = 229977225$, $(21135)^2 = 446688225$, and $(29685)^2=881199225$.

No. 241. Proposed by *W. V. Parker*, Louisiana State University.

In a circle, a diameter $AB=d$ is extended one unit to C. The tangent at A and the tangent from C meet at P with $AP=t$. Determine all integral values of the pair (d,t) and show that for each one PC is also integral.

Solution by *C. W. Trigg*, Los Angeles City College.



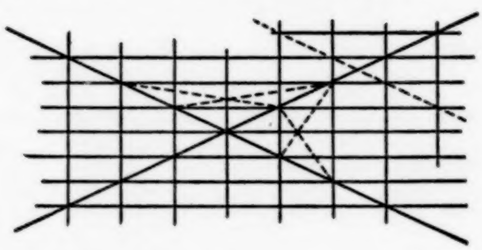
Let M be the point of contact of PC, and O be the center of the circle. Then $PM=t$, $OB=OM=d/2$ and $MC=\sqrt{(d/2+1)^2-(d/2)^2}=\sqrt{d+1}$. Now triangles PAC and OMC are similar, so $\frac{1}{2}d : t :: \sqrt{d+1} : (d+1)$. Whence, $d\sqrt{d+1}=2t$. Set $d=m^2-1$, then $t=\frac{1}{2}m(m^2-1)$ and $PC=t+\sqrt{d+1}=\frac{1}{2}m(m^2+1)$, which gives a one parameter solution for (d,t,PC) with m any positive integer. t is integral since the product of three consecutive integers, $(m-1)m(m+1)$, is always divisible by 2, and since $\sqrt{d+1}=m$, then PC is also integral .

Also solved by *Walter B. Clarke* and *Fred Marer*.

No. 243. Proposed by *Robert C. Yates*, University of Maryland.

Given a straight edge with two marks on it. With this instrument alone construct a rectangular network of lines.

Solution by *C. W. Trigg*, Los Angeles City College.



With the straight-edge draw two intersecting lines. Starting from the point of intersection, these two lines may have any desired number of equal segments marked off on them by means of the two marks. Connect the first point of division of one of the lines to the second point (or points) on the other. The lines through the vertex and the intersections of these lines bisect the angles formed by the original lines. These together with the lines through the points equidistant from the vertex form a rectangular network of lines. If the two bisectors be taken as the two fundamental lines, a square network is obtained. When the distances between the points of the fundamental lines become greater than the length of the straight edge, the diagonals of one or more of the extreme rectangles may be drawn and the method continued with them.

Also solved by *Walter B. Clarke* and the *Proposer*.

PROPOSALS

No. 259. Proposed by *Walter B. Clarke*, San Jose, California.

Construct a triangle whose verbicenter lies on its incircle.

No. 260. Proposed by *M. S. Robertson*, Rutgers University.

Show that the area of a triangle formed by three tangents to a parabola is one-half the area of the triangle formed by the three points of tangency.*

No. 261. Proposed by *V. Thébault*, Le Mans, France.

Determine a perfect square corresponding to each of the following forms:

(1) *abc*578*abc*, (2) *abcabc*984, (3) 456*abcabc*.

No. 262. Proposed by *V. Thébault*, Le Mans, France.

Form a perfect square, the product of whose six digits is 190512.

No. 263. Proposed by *C. N. Mills*, Illinois State Normal University.

Given the sequence of terms

$$(n^2-2n+2),\quad (n^2-4n+6),\quad (n^2-6n+12),\quad (n^2-8n+20),\quad (n^2-10n+30),\ldots,$$

determine values of n such that each of the first $(n-1)$ terms of the sequence is prime.

No. 264. Proposed by *E. C. Kennedy*, Texas College of Arts and Industries.

Let
$$T_n=\sqrt[3]{5+2T_{n-1}},\qquad T_0=\sqrt[3]{5},$$
$$S_n=\sqrt{(5+2S_{n-1})/S_{n-1}},\qquad S_0=\sqrt{2}.$$

Prove that:
$$\lim_{n\to\infty} T_n=\lim_{n\to\infty} S_n.$$

No. 265. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

Prove (or disprove): The sum of the pth powers of the distances to the sides of a regular $n-$gon from a point on its incircle is constant

*Probably well known—but is it available in print?—Ed.

with regard to the position of the point, provided p is a positive integer less than n. This constant is given by

$$\sum_{i=1}^{i=n} |d_i|^p = \frac{1\cdot 3\cdot 5\cdot \ldots (2p-1)}{p!}\cdot n\cdot r^p,$$

where r is the inradius of the n-gon.*

No. 266. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

Find all positive integral values of n which satisfy the equation:†

$$1^n+2^n+3^n+2^{2n-1}+(2+\sqrt{3})^n+(2-\sqrt{3})^n \pm \frac{1\cdot 3\cdot 5\cdot \ldots (2n-1)}{n!}\cdot 2^{n+1}\cdot 3.$$

*See the additional discussion under Problem 3774, *American Mathematical Monthly*, March, 1938.

†In connection, see Problem 176, this Magazine, February, 1938.